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CLOSED ALGEBRAIC CORRESPONDENCES.

By Albert A. Bennett.

Correspondences and their Reducibility.

1. Let M be a complex algebraic manifold of one dimension. We may regard M either as a complex curve in a space in which the curve is without multiple points, or we may think of M abstractly as a Riemann surface not only without multiple points but without recognizable branch points and points "at infinity." We shall use the letters x, y, $\cdots$, to denote representative points on the manifold, although, of course, it is impossible to identify a point of M by means of a single *rational* parameter except when M is itself rational. We shall write $C(x, y)$ as the symbol* of an algebraic correspondence on M, by which for any given point x of M, there are algebraically determined n points y of M, in general distinct, regarded as corresponding to x, all of which vary as x varies, and conversely, for any given y, there are determined algebraically m points x of M, in general distinct, each of which yields the given y by means of the correspondence, and each of which varies with y. These numbers, m and n, which are called the *orders* of the correspondence, may be brought more explicitly into evidence by writing the symbol in the form $C(\overset{m}{x}, \overset{n}{y})$. Any analytic correspondence with no essential singular points is readily seen to be of finite orders, or in other words, algebraic, cf. Hurwitz, l. c.

For a given x we obtain n and only n values of y by means of $C(x, y)$. We may let x make any desired circuit on M, but whenever x returns to its initial position, the set of values of y must return to its initial position. Permutations may however have occurred within the set. The totality of all permutations that can be obtained by means of closed circuits described by x will form a permutation group G of degree n but of order depending upon the correspondence. We may select an initial position ξ_0 for x, for which no two of the corresponding values of y coincide, and select for $x = \xi_0$ one of the points y which we shall call y_1. As x makes all types of closed circuits starting from ξ_0 and returning to it, it may or may not happen that y_1 is carried into each of the other y's, that is, it may or may not happen that G is transitive. If G is non-transitive, then C is said to be *reducible* with respect to y, but otherwise, *irreducible* with respect to y. We may similarly define irreducibility with respect to x.

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* When in place of M, we take a plane curve defined by an equation $f(x_0, x_1, x_2) = 0$ where x_0, x_1, x_2 are homogeneous, there exist correspondences which can be defined as the common solution of two equations, $P_1(x_0, x_1, x_2;\ y_0, y_1, y_2) = 0$ and $P_2(x_0, x_1, x_2;\ y_0, y_1, y_2) = 0$ on $f(x_0, x_1, x_2) = 0$, but which cannot be expressed by means of a single equation $P(x_0, x_1, x_2;\ y_0, y_1, y_2) = 0$, on $f = 0$. Cf. Hurwitz, Math. Ann., 28, 561 ff. (1887).

Addition and Multiplication of Correspondences.

2. We may consider certain methods of combining correspondences which may be denoted by addition and multiplication respectively.* Indeed, if $C_1(\overset{m}{x}, \overset{n}{y})$ be one given correspondence and $C_2(\overset{p}{x}, \overset{q}{y})$ be another, then by $C_3 = C_1 + C_2$, will be meant the correspondence $C_3(\overset{m+p}{x}, \overset{n+q}{y})$ obtained by regarding every point y corresponding to the x supposed given, whether by C_1 or by C_2 as corresponding to this x by C_3, and hence similarly with the x's yielding a given y. We may introduce the symbol, zero, 0, to stand for any "correspondence" which involves neither x nor y. Hence $C + 0 = 0 + C = C$. Further, $C_1 + C_2 = C_2 + C_1$, $C_1 + (C_2 + C_3) = (C_1 + C_2) + C_3$, or in other words, addition of correspondences is commutative and associative.

In an analogous manner we shall define the product $C_3 = C_1C_2$, as the correspondence $C_3(\overset{mp}{x}, \overset{nq}{y})$ obtained from C_1 and C_2 by starting with a given x and obtaining the n z's given by $C_1(\overset{m}{x}, \overset{n}{z})$ and then for each of these z's, finding the q y's given by $C_2(\overset{p}{z}, \overset{q}{y})$, and finally regarding these nq y's as the y's corresponding to the given x by C_3. This will also give us the mp x's for a given y obtained by considering the converse relationships. The correspondence given by $y = x$, will be the identity for multiplication and may be denoted by the symbol 1. The symbol aC^b, where a and b are given positive integers will now have a definite meaning for any given C. The distributive character of multiplication is readily verified.

If C_1 be a correspondence irreducible with respect to y and another different correspondence, C_2 be given such that as x varies, one of the points y obtained from x by C_1 is always obtainable from x also by C_2, then C_2 contains all of the points corresponding to x by C_1, since a suitable closed circuit of x will carry this common y into any other desired point y given by C_1.

If C^2 contains x we shall have an equation of the form

$$C^2 = a + C_1,$$

where a is a positive integer indicating the number of times C^2 contains x.

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* Cf. Severi, Torino, Mem. (2), 54, 1 (1903).

If C^2 contains any point already given by C, we shall have

$$C^2 = bC + C_2,$$

where b is the number of times C^2 contains C. It often happens that there exists an integer k, such that C^k can be expressed entirely in terms of C's of lower powers. In such a case there are k integers, $a_1, a_2, \cdots, a_k$, such that

$$C^k = a_1C^{k-1} + a_2C^{k-2} + \cdots + a_{k-1}C + a_k.$$

These cases are particularly interesting as the C's can then give only a finite number of distinct points by any amount of addition and multiplication, and a sort of finite number system is constructed. For any C satisfying an equation of the above form, we must have the orders m and n, equal. For, in terms of the orders we have

$$m^k = a_1m^{k-1} + a_2m^{k-2} + \cdots + a_{k-1}m + a_k,$$
$$n^k = a_1n^{k-1} + a_2n^{k-2} + \cdots + a_{k-1}n + a_k.$$

Thus m and n would have to be, otherwise, distinct real positive roots of an equation which by Descartes' rule of signs can have at most one real positive root. Hence $m = n$. There exist, also, obvious restrictions upon the integer coefficients a_i, $i = 1, 2, \cdots, k$ if there is to be a solution m for the order of the correspondence.

Closed Correspondences on an Algebraic Manifold M.

3. We have mentioned a type of case in which the total number of points obtained from a variable point x, by means of $C(x, y)$ is finite. We shall introduce the more general notion of a closed system as follows: If a correspondence $C(x, y)$ be such that starting from a variable point x, the maximum number of distinct points given by any correspondence of the form, $1 + C + C^2 + \cdots + C^h$ is finite and equal to ν, no matter how great h be taken, nor how x varies over the manifold M, then C is said to generate a closed system.

Let us suppose we have generated a closed system S of ν points on M. These ν points will form a one-parameter involution of index one,* since the entire system as a whole, although not the relative positions of the points among themselves, is fixed uniquely by the position on M of any point of the set, without regard to distinctions among the points. We shall now examine some properties of this type of involution.

We are given a set of ν variable points, among which there will be, in particular, a permutation group G' consisting of the permutations of the remaining $\nu - 1$ points, which arise when a selected point varies over closed circuits on M. This group will be independent of the particular variable point selected and of the fixed position ξ_0 on M from which this variable point starts out and to which it returns, provided only that at most a finite number of special points which serve as branch points for the group be excluded from M. Let μ be the order of G'. Let x be a point of S, and let $r_1, r_2, \cdots, r_e$ be a set of independent rational functions of x and of the remaining $\nu - 1$ points of S, these functions being unsymmetric in the points of S. The value of r_i, $i = 1, 2, \cdots, e$, will not be determined by x alone but will have μ different determinations for a given position of x, corresponding to the permutations of the remaining points of S. In any given case we may take e sufficiently large so that the values of the e r's being known, the point x is completely determined, although in every case x will determine μ sets of values of the r's. By using the r's as the parameters of a point in e-space, we define a one-dimensional algebraic manifold M' such that each point x of M determines algebraically μ points of M' in general distinct, and each point of M' is obtained from but a single point of M. We may then regard M' as mapped as a μ-sheeted Riemann surface upon M.

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* Enriques, Rend. Circ. M. Palermo, 10, 30 (1895).

A One-to-One Reciprocal Correspondence Group on the Associated Manifold M'.

4. Let the ν points of the involution S on M, be numbered arbitrarily as, $x_1, x_2, \cdots, x_\nu$, and let the μ points of M' corresponding to the point x_i of M, be numbered arbitrarily as $x_{i1}, x_{i2}, \cdots, x_{i\mu}$. Then if we observe that a point, for example, x_{11}, of M' corresponds not merely to a point, viz., x_1, of M, but also to a definite permutation among the other points, $x_2, x_3, \cdots, x_\nu$, of M, we see that whenever the point x_{11}, describes a closed circuit, each of the points, $x_1, x_2, \cdots, x_\nu$, and therefore also each of the sets $(x_{i1}, x_{i2}, \cdots, x_{i\mu})$, $i = 1, 2, \cdots, \nu$, is transformed into itself. Moreover since the points, $x_1, x_2, \cdots, x_\nu$, of the involution are not permuted, and since the points within a set $(x_{i1}, x_{i2}, \cdots, x_{i\mu})$, correspond to the different permutations among the points of the involution, the points within a set $(x_{i1}, x_{i2}, \cdots, x_{i\mu})$ are also carried individually into themselves. Let S' be the set of variable points x_{ij} on M', $\mu\nu$ in number. The correspondence on M' between any one point of S' and any other is therefore not only algebraic but actually one to one reciprocally. Hence S' consists of a set of points obtained by a group of one-to-one correspondences on M', and M' admits of a group of algebraic one-to-one reciprocal correspondences into itself.

Conditions upon Correspondences which Generate Closed Systems.

5. Interpreted algebraically, the connected surface M' which is μ-sheeted with respect to M, determines an irreducible algebraic irrationality

A on M, such that given any point x of M, $x' = A(x)$ is an irreducible μ-valued algebraic function of x, whose array of values may be used to construct M' upon M. We may indicate the μ determinations of A by subscripts, thus $A_j(x)$, $j = 1, 2, \cdots, \mu$ although two of these determinations may be permuted by a closed circuit on M. Given x and an associated value of x', we have always but a single point of M' determined. The fact that all the points x_{ij} vary over the same manifold M' is equivalent to saying that given any x_{ij}, that is given x_i and its associated $x_{ij}' = A_j(x_i)$ any other $x_{i'j'}$ is birationally determined, that is both $x_{i'}$ and its associated $x'_{i'j'} = A_{j'}(x_{i'})$ are birationally determined.

In terms of the original correspondence $C(x, y)$, this gives us the following: A necessary condition that a correspondence $C(x, y)$, irreducible with respect to y, can generate a closed system is that there exist on M, an irreducible algebraic μ-valued function A, such that given x and any desired determination of $A(x)$, say x', each y satisfying $C(x, y)$ together with each associated $y' = A(y)$ is obtainable rationally, while this pair of values, (y, y'), also determine x and x', rationally. We may represent $C(x, y)$ symbolically by a set of equations of the type,

$$y = \varphi_1(x, x'), \qquad x = \psi_1(y, y'),$$
$$y' = \varphi_2(x, x'), \qquad x' = \psi_2(y, y'),$$

where φ_1, φ_2, ψ_1, ψ_2, are all rational for $x' = A(x)$, and $y' = A(y)$.

If M' is of genus greater than one, which often happens when the genus of M is zero or one, and always when the genus of M is greater than one, the above condition is also sufficient, since in this case the entire system of points will be generated by a group of one-to-one reciprocal correspondences on a one-dimensional algebraic manifold, which is in every instance a finite group.* For M' of genus zero or one, we must introduce additional conditions of closure if the set is required to be finite. These conditions are algebraically expressible, but occur very naturally as linear relations among rational or elliptic functions according as M' is of genus zero or one. The best known examples are furnished by the closure conditions for Poncelet and Steiner polygons.†

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* Various numerical relations between the genera of manifolds related as M and M' are known. Cf., e. g., Zeuthen, Lehrbuch der abzählenden Methoden der Geometrie, Chap. IV, Teubner, 1914. That the group of transformations for a genus greater than one is finite was proved most elegantly, perhaps, by Hurwitz, Math. Ann., 32 and 41, but many earlier and later proofs have been made. Cf. Pascal, Repertorium, II, 1, 2d ed., p. 352.

† For an explicit algebraic discussion and derivation of the algebraic conditions, cf. "An algebraic treatment of the theorem of closure," by the author, *Annals of Mathematics* (16), p. 96 (1915).

Groups Associated with Closed Correspondences.

6. The system S' of points x_{ij} on M' constitute a set of points obtainable one from another by a group of one-to-one reciprocal correspondences. We may consider these one-to-one correspondences as elements of a group H of operations permuting the points of S'. Among the permutations of H, will be those carrying a given point, e. g., x_{11}, regarded as initial, into other points of the same set $\{x_1\}$. The permutations leaving $\{x_1\}$ invariant form as we have seen a group G' of μ operations. When we take into consideration the manifold M, we see that the group G' leaves invariant the point x_1 on M although causing permutations among the other points. If the entire system S on M is to be generated by a single correspondence irreducible in y, we must have on M' a permutation s which carries x_{11} into some point x_{ij}, and which when multiplied by the operations of G' carries x_1, regarded as a given point x, into each of the points, y, obtained by the correspondence, $C(x, y)$. The permutation s together with the operations of G', will suffice to generate the entire system S' on M' and its image S on M. Thus H may be generated by means of the subgroup of operations $\{t\}$ of G', together with a single additional operation s not of G'. We shall say that H is generated irreducibly by s with respect to G'. If for every operation u of H and for every t_1 of G' there were another operation t_2 of G', such that $ut_1 = t_2u$, that is, if G' be a self-conjugate subgroup of H, then the same point of M would be determined no matter from what sheet of M' we started and the construction for M' would make this manifold a duplicate of M. Hence G' is not a self-conjugate subgroup of H.

It sometimes happens that the system S corresponds to a set of one-to-one transformations on M and that G' is a set of outer isomorphisms* carrying S into itself and carrying the operation s into a complete set of generators. This, although an important case is not the only one as is seen even in the simple instance of the Poncelet polygons, unless we regard the group from an abstract and non-algebraic point of view by introducing such notions as sense.

For any group H, irreducibly generated with respect to a non-self-conjugate subgroup G', we may construct a manifold M', and a manifold M with a closed correspondence system. The fundamental region of M', by means of duplicates of which M' may be constructed as a whole, is to a very large extent arbitrary,† and this degree of arbitrary choice remains, of course, in the manifold M, on which the correspondence is determined. Almost the entire theory of abstract groups and the additional theory incident to the particular representation of a group as irreducibly generated, as well as the theory relating to the nature of the fundamental region is reflected in the study of closed correspondences. We shall next turn to a particularly important application of the theory of closed correspondences.

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* Cf. Burnside, Theory of Groups, 2d ed., p. 84.

† Cf. A. Hurwitz, l. c.

Variable Inscribed Plane Configurations.

7. A system of distinct points $\{P\}$, and lines $\{l\}$, such that on each point P there lie b of the lines $\{l\}$, and on each line l, there lie a of the points $\{P\}$, where a and b are fixed finite numbers, and independent of the particular points and lines selected, will be called a *configuration.** A configuration is *finite* whenever the total number of points and lines is finite. Any non-finite configuration must, of course, present singularities when viewed externally, and for this reason only finite configurations are usually considered. The case in which a Poncelet polygon does not close, but becomes an infinite broken line affords a simple illustration of a configuration that is not finite. Any two configurations which are in one-to-one correspondence so far as incidence relations among the given elements are concerned will be regarded as equivalent, irrespective of additional projective properties of the separate configurations. Configurations as here defined may be referred to as plane configurations in distinction from the figures in higher dimensional spaces obtained by an obvious generalization of the above definition.

Let us be given a particular configuration in the projective plane. It is clearly possible to pass an irreducible curve through all of the points of the configuration, and this, too, in an infinite number of ways, provided only that the degree of the desired curve be not previously restricted. The configuration may be said to be *inscribed* in any such curve. It sometimes happens that the curve may be so chosen in a fixed position that a certain number r, > 0, of points may be selected arbitrarily on the curve and an entire configuration equivalent to the one initially given may be constructed having the r points as points of the configuration and also, like the given one, being inscribed in the curve. The configuration may then be regarded as *varying on the curve with r degrees of freedom.*

Inscribed configurations of more than one degree of freedom present certain special problems of interest, with which, however, we shall not be concerned at this time. From the point of view of the configuration itself, the number of degrees of freedom is of relatively little significance. When there is more than one degree of freedom, we may always introduce algebraic restrictions so as to cut down the degrees of freedom to one, and it is this case, exclusively, that we shall consider.

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* Cf. article in Encyc. des Sc. Math., E. Steinitz and E. Merlin, III, 9.

The Index of a Configuration, Total and Partial Configurations.

8. Let us be given an irreducible curve M, and a variable configuration K, inscribed in it, and having by virtue, perhaps, of additional conditions, one and but one degree of freedom. Let P be a point of K, and P_0 and K_0 be particular positions of P and K for which K_0 contains P_0. As P varies over M from P_0, K varies from K_0. If every possible circuit of P starting from and returning to P_0 carries the configuration K as a whole into the position K_0, then K is said to be of *total index*, *one*. If, however, there be τ different positions, $K_{0(1)} \equiv K_0$, $K_{0(2)}$, $\cdots$, $K_{0(\tau)}$, into which K can be carried when P returns to P_0, then the configuration is said to be of total index τ. Since the positions of the various points of the configuration are determined algebraically from the position of one variable point, P, when the position P_0 has been chosen, the remaining choices contain no free parameter and are obtainable algebraically, so that for a representative choice of P_0, τ is a finite positive integer. We shall confine ourselves to variable configurations of index one. It is readily seen that a configuration of index greater than one, generates a finite or infinite configuration of index one.

If we be given a variable, plane, inscribed configuration K, on a curve M, of one degree of freedom, and of total index one, then every point of the configuration is on one of the lines of the configuration and is a point of M, but a converse, that every point of intersection with M, by a line of the configuration, is a point of the configuration, does not follow. We shall usually be obliged to distinguish certain points not of the configuration, but on M and also on one or more of the lines of the configuration, which are fixed as the configuration varies. These will be called fixed points or base points of the configuration system. There may or may not be additional points of M on lines of the configuration, which, however, are not points of the configuration. These may be called, when existent, *extraneous variable points* of the system. When no extraneous variable points exist we shall have what we shall call a *total configuration*, otherwise, a *partial configuration*.

Configurations and Correspondences.

9. A variable configuration K, on a curve obviously establishes a closed correspondence system S on the curve, of a specially simple sort. When the configuration is of total index, one, the closed correspondence is an involution, also of index, one. The correspondence will be irreducibly generated when the following conditions are satisfied: First, if for any point P of the configuration, and a suitably chosen point P_1 on a line of the configuration containing P, we may, by a suitable circuit

on M, carry P back to its initial position and the line PP_1 into every other line of the configuration containing P. Secondly, if the correspondence from P to P_1, when repeated sufficiently often, will carry P into every one of the points of the configuration on the line PP_1. We therefore have the theorem that any irreducibly generated, variable, inscribed configuration with one degree of freedom on a curve of genus greater than one is necessarily finite.

The question arises as to whether every closed correspondence S may be interpreted as a configuration K. For various reasons the partial* configurations do not have the same interest as the total configurations. It is these latter that we shall consider.

We notice, first, that if we be given a total configuration K, the lines of K are a subset of the total number of lines of the plane, and the set of points of K, on a line of K, may be regarded as a special case of the points of intersection of an arbitrary line of the plane with the curve. As the configuration varies on the curve we shall, in any case, have a one-parameter family of lines which is a subset of the total two-parameter family, this subset being determined algebraically. Now let L be the set of points obtained by the given correspondence C, as the images on M of the set generated on M' by s alone. If C be such that L varies within a two-parameter linear family of the same order as L, and varies in such a way that no two distinct sets L of the two-parameter family contain on M as many as two common points, then we may so represent M as a curve, that S becomes the system of points in a total inscribed variable configuration K. For, by the usual theory of birational transformations, we may represent the sets L as cut out by straight lines, by means of a birational transformation, since no two distinct sets L have more than a point in common.† The set of lines in the system S are determined algebraically, giving us a line equation for a locus whose tangents determine the correspondence. The conditions that no two L-sets have more than one point of M in common and that they vary in a two-parameter linear family of the same order as L, are thus both necessary and sufficient that we have S representable as a total, variable, inscribed configuration K with one degree of freedom.

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* The partial configurations are related to irrational involutions, and the total, to rational involutions.

† Severi, Lezioni, p. 18.